

Korteweg–de Vries Positive and Negative Hierarchies

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(7 July 2003)

We present an approach to the construction of action principles (the inverse problem of the Calculus of Variations), for first order (in time derivatives) differential equations, and generalize it to field theory in order to construct systematically, for integrable equations which are based on the existence of a Nijenhuis (or hereditary) operator, a (multi-Lagrangian) ladder of action principles which is complementary to the well-known multi-Hamiltonian formulation. We work out results for the Korteweg–de Vries (KdV) equation, which is a member of the positive hierarchy related to a hereditary operator. Three negative hierarchies of (negative) evolution equations are defined naturally from the hereditary operator as well, in a concise way, suitable for field theory. The Euler–Lagrange equations arising from the action principles are equivalent to deformations of the original evolution equation, and the deformations are obtained explicitly in terms of the positive and negative evolution vectors. We recognize, after appropriate coordinate transformations, the Liouville, Sinh–Gordon, Hunter–Zheng and Camassa–Holm equations as negative evolution equations. The multi-Lagrangian ladder for KdV is directly mappable to a ladder for any of these negative equations and other positive evolution equations (e.g., the Harry–Dym and a special case of the Krichever–Novikov equations). For example, several nonequivalent, nonlocal time-reparametrization invariant action principles for KdV are constructed,

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and a new nonlocal action principle for the deformed system Sinh–Gordon + spatial translation vector is presented. Local and nonlocal Hamiltonian operators, are obtained in factorized form as the inverses of all the nonequivalent symplectic 2–forms in the ladder. Alternative Lax pairs for all negative evolution vectors are constructed, using the negative vectors and the hereditary operator as only input. This result leads us to conclude that, basically, all positive and negative evolution equations in the hierarchies share the same infinite–dimensional sets of local and nonlocal constants of the motion for KdV, which are explicitly obtained using symmetries and the local and nonlocal action principles for KdV.

Keywords: Multi–Lagrangians, Hereditary Operators, KdV Hierarchies, Lax Pairs.

Pacs Numbers: 02.30.Zz, 02.30.Ik, 11.30.-j, 11.10.Lm .

I. INTRODUCTION

Hereditary or Nijenhuis operators^{?,?,?} play an important role in the description of integrable systems: in terms of these operators, the very definition of the positive and negative hierarchies of integrable evolution equations may be given,[?] and they are used to construct, for these equations, symmetries,^{?,?} constants of the motion,[?] alternative Lax pairs,[?] multi-Hamiltonian structures[?] and weakly-nonlocal multi-symplectic and multi-Hamiltonian structures.^{?,?,?}

The problem of constructing multi-Lagrangian structures (i.e., an infinite ladder of action principles) for the Korteweg-de Vries (KdV) equation has been tackled recently^{?,?} in the context of localizable multi-Lagrangian structures, using the bi-Hamiltonian formulation. However, the explicit expression of the action principles associated to each symplectic 2-form requires the integration of the respective 2-forms, a task which is increasingly difficult as we move to the positive end of the ladder, because of the increasing complexity of the differential terms of the (weakly-nonlocal[?]) 2-forms. On the other hand, the symplectic 2-forms are increasingly nonlocal as we move to the negative end of the ladder. The known way^{?,?} to get rid of the nonlocality problem is to write the action principles in a “local” coordinate system (Darboux coordinates) depending on the specific symplectic 2-form in the ladder, a process that gets recurrently harder as we move to the negative end. Then, again, the 2-form must be integrated by hand in order to get the action principle.

In this work, we make use of the Galilean symmetry[?] and the factorized form of the hereditary operator[?] for the KdV equation, to construct explicitly the action principles for KdV in the positive and negative parts of the ladder. No integration of any 2-form is needed, nor is the search for a special coordinate system. The factorized form of the symplectic 2-forms allows for the interpretation of the resulting Euler-Lagrange equations (arising from each action principle) as deformed equations, with flows given by KdV + vectors in the positive and negative hierarchies, which are computed explicitly.

Explicit expressions for local and nonlocal constants of the motion for KdV are obtained using symmetries along with the local and nonlocal action principles.

From the action principles obtained for the KdV equation we construct action principles for flows defined by other positive and negative vectors. In particular, a new nonlocal action principle for the Sinh-Gordon (ShG) equation[?] (a negative equation) are constructed.

It is a known result[?] that alternative Lax pairs for the KdV equation and for positive KdV

flows may be constructed from the hereditary operator. Here we do the same construction for all the negative KdV flows, and we conclude that the local and nonlocal constants of the motion for KdV, define conserved currents and constants of the motion for all the negative flows as well.

The results here may be mapped to the following equations: a special case of Krichever–Novikov,^{?,?} Harry–Dym,[?] Camassa–Holm,[?] Hunter–Zheng,[?] ShG[?] and Liouville, all of which are essentially flows belonging to the KdV positive or negative hierarchies. We stress that the results are quite general and may be extended to other systems related to hereditary operators (e.g., nonlinear Schrödinger equation).

This paper is organized as follows: section II presents a preview and notation for the method of construction of action principles for given differential evolution equations, and a brief survey of symmetries and constants of the motion in this context. Next, we show the relationship of these Principles with Hamiltonian theories, and finally we introduce the hereditary property with the consequent construction of the positive and negative hierarchies of integrable evolution equations. In section III we present and prove theorems on the explicit construction of ladders of action principles and constants of the motion, based on the hereditary operator and the Galilean symmetry (whose definition is quite general and not restricted to KdV), and we show how these action principles give rise to Euler–Lagrange equations which are deformations of the original equations, due to the fact that the symplectic operators have a nonzero kernel.

Section IV is devoted to examples of the above constructions for the KdV equation. We obtain concise expressions for the negative vectors, for the action principles, and for the deformed Euler–Lagrange equations. Symplectic operators are presented in factorized form (this allows for factorized expressions for the Hamiltonian operators). Some known integrable evolution equations are identified within the negative hierarchies. Nonlocal constants of the motion for KdV are concisely obtained using the internal symmetries and the negative action principles. In section V we work out examples of new nonlocal action principles for the ShG equation, which in this context is identified as a negative vector; we construct Lax pairs for the negative equations, thus showing that the local and nonlocal constants of the motion for KdV also work for negative equations. Finally, some concluding remarks are presented in section VI.

For simplicity, we work in finite-dimensional notation. All assertions and theorems in

sections II and III are valid in finite dimensions, and they can be extended to the case of field theory in all the instances dealt with in this paper. We have used a Mathematica code of our invention, to confirm the validity of some of the obtained local and nonlocal action principles, symmetries and constants of the motion for KdV and related equations.

II. PREVIEW AND NOTATION

Consider the autonomous equations of motion

$$\dot{q}^a(t) = V^a[q^b(t)], \quad a \in A. \quad (1)$$

A is a given ordered set called “label set”: the elements of it label the degrees of freedom of the theory. From now on, we suppress the dependence of the coordinates $\{q^a\}_{a \in A}$ on time when it is obvious.

Example 1. The KdV Equation for the field $u(x, t)$, $x \in [x_-, x_+]$, $t \in \mathbb{R}$, is

$$u_t = -u_{xxx} - 12 u u_x \quad (2)$$

(suffixes denote partial differentiation). The label set is $A = [x_-, x_+]$, and $x \in A$ is a continuous index.

We will use standard boundary conditions for the field: $u, u_x, \dots \rightarrow 0$ as $x \rightarrow x_{\pm}$, and we will set $x_{\pm} = \pm\infty$, although the methods may be extended for the treatment of other boundary conditions as well (in which case the Weiss action principle[?] and the Witten–Zuckerman 2–form[?] come into play).

The evolution equation (1) is naturally defined on a vector space spanned by the derivatives $\{\frac{\partial}{\partial q^a}\}_{a \in A}$ (for the infinite–dimensional case, partial derivatives with respect to the coordinates become functional derivatives). We call $V = V^a \frac{\partial}{\partial q^a}$ the **flow** vector or evolution vector for the system (1), where here, and throughout in this paper, Einstein summation convention over repeated indices is assumed (for the infinite–dimensional case, the summation is extended to an integration over continuous indices).

A. Action Principles

The equations of motion (1) are related to a **Variational Principle** with **Action**

$$S[q^a(t), t] = \int_{t_-}^{t_+} dt \left(P_a (\dot{q}^a - V^a) + K \right), \quad (3)$$

where the 1-form $P[q^b]$ and the zero-form $K[q^b]$ satisfy the following equation:

$$P_{a,b}V^b + P_bV^b_{,a} = K_{,a},$$

with $K_{,a} \equiv \frac{\partial K}{\partial q^a}$.

We rewrite the above equation in terms of invariant structures:

$$\mathcal{L}_V P = \delta K, \quad (4)$$

where \mathcal{L}_V is the **Lie derivative** along the vector V , and δ is the **exterior differential** (see[?] for a definition of these operators).

Definition 2. We call the pair $(P; K)$ a **standard Lagrangian pair** for V if $K \neq 0$. In the special case $K = 0$ we call P a **non-standard Lagrangian** (1-form) for V : the latter case allows for the construction of constants of the motion in a direct way[?] (see theorem 9).

Remark 3. The above objects should not be confused with the usual “Lagrangian density” $L[q, \dot{q}, t] = P_a(\dot{q}^a - V^a) + K$, which is the thing that is integrated in time to give the action: $S = \int L dt$. The 1-form P is also understood as a momentum map.[?] When P is a non-standard Lagrangian, it solves the equation for a “conserved covariant.”[?]

The general case of objects which depend explicitly on time is easily worked out,[?] but there is no need to do so in the applications of this paper. Nevertheless, for symmetries and constants of the motion the explicit time dependence will be necessarily taken into account. In the sequel, we give the name **time-(in)dependent** to those objects which do (not) depend **explicitly** on time.

The **Euler–Lagrange** equations which come from the action (3) are:

$$\Sigma_{ab}(\dot{q}^b - V^b) = 0,$$

where $\Sigma \equiv \delta P$ is the **symplectic 2-form** or Lagrange bracket whose components are:

$$\Sigma_{ab} = P_{b,a} - P_{a,b}.$$

It is worth mentioning that, in[?], a symplectic 2-form is induced by an action principle in essentially the same way we have derived the above symplectic 2-form from the action principle (3).

Notice that these Euler–Lagrange equations do not imply the original equations of motion (1); instead they imply **deformed** or mixed equations, where the deformation is represented

by an additive extra term which is an arbitrary linear combination of vectors belonging to the **kernel** of the symplectic 2-form. In the case of KdV, we will obtain the deformations explicitly. See⁷ for examples in the finite dimensional case.

The symplectic 2-form associated to this action principle is easily seen to satisfy

$$\begin{aligned}\delta \Sigma &= 0 \quad (\text{closure}) \\ \mathcal{L}_V \Sigma &= 0, \end{aligned} \tag{5}$$

therefore the inverse process could be done: starting from a symplectic 2-form Σ for the flow vector V , we construct the standard Lagrangian 1-form, from $\delta P = \Sigma$ and the 0-form K is obtained by integration of equation (4). This process suffers from technical difficulties, which increase when the objects are infinite dimensional and nonlocal. Fortunately, for the KdV equation there is a constructive way of finding the action principles (see theorems 6 and 7).

B. Hamiltonian theories are induced from Symplectic Structures

The relationship of the symplectic 2-form with the Hamiltonian formulation is very simple:⁷ consider the formal inverse (i.e., except for a finite kernel that the operators may possess) of the above 2-form, the time-independent $(2, 0)$ tensor J such that $J \cdot \Sigma = \mathbb{I}$. It is possible to show that Σ is closed if and only if J satisfies the **Jacobi identity**, which we write in the form:

$$\mathcal{L}_{J \cdot U} J = J \cdot \delta U \cdot J, \quad \forall \text{ 1-form } U. \tag{6}$$

J is known as a Hamiltonian operator or Poisson bracket. Now, equation (5) implies $\mathcal{L}_V J = 0$. Therefore, according to the Jacobi Identity (6), a Hamiltonian theory for the flow V is induced by the symplectic 2-form Σ : the equation $V = J \cdot U$ implies $\delta U = 0$, thus $V = J \cdot \delta H$, where H is the Hamiltonian, a time-independent 0-form.

C. Symmetries

Symmetries play a crucial role in the construction of the action principles. A symmetry for the system (1) is known as a vector with components η^a that takes solutions into solutions of equation (1), in the sense that given any solution $q^a(t)$ such that $\dot{q}^a(t) = V^a[q^b(t)]$, then $\tilde{q}^a \equiv q^a + \epsilon \eta^a[q^b, t]$ is also a solution up to order ϵ^2 , i.e.: $\dot{\tilde{q}}^a(t) = V^a[\tilde{q}^b(t)] + O(\epsilon^2)$.

It is easily seen[?] that this condition leads to the equation $\frac{\partial}{\partial t}\eta^a + \eta^a_{,b} V^b - V^a_{,b} \eta^b = 0$ or, in a covariant way, $\left(\frac{\partial}{\partial t} + \mathcal{L}_V\right)\eta = 0$.

Example 4. The Galilean and the dilatation symmetries for the KdV equation are defined respectively by

$$\begin{aligned}\eta_G[u, t]^x &= \frac{1}{8} - \frac{3}{2} t u_x, \\ \eta_D[u, t]^x &= u + \frac{1}{2} x u_x - t \left(\frac{3}{2} u_{xxx} + 18 u u_x \right).\end{aligned}\tag{7}$$

In[?], there is an open question concerning the role of the Galilean and dilatation symmetries in the construction of constants of the motion for KdV. An answer to this question is given in this work: these symmetries actually lead to action principles for the KdV equation, which are involved in Noetherian and non-Noetherian constructions[?] of constants of the motion (see theorems 6, 7, and 9).

D. Constants of the Motion

A constant of the motion for the system (1) is a functional (0-form) $C[q^a, t]$ which is conserved in time under the evolutionary system: $\frac{D}{Dt}C[q^b, t]|_{\text{on-shell}} \equiv \frac{\partial}{\partial t}C + C_{,a} V^a = 0$, where the partial time derivative accounts for the explicit time dependence and $\frac{D}{Dt}$ denotes the convective or total derivative along the variable t .

This equation is best written in a covariant way:[?] $\left(\frac{\partial}{\partial t} + \mathcal{L}_V\right)C = 0$.

We will usually work with time-independent constants of the motion: $\frac{\partial}{\partial t}C = \mathcal{L}_V C = 0$.

E. The Hereditary Property: Hierarchies of Evolution Equations

Many integrable systems are related to a Nijenhuis or hereditary operator, which is a time-independent $(1, 1)$ tensor R that solves:[?] $\mathcal{L}_{R \cdot \eta} R = R \cdot \mathcal{L}_\eta R$, \forall vector η .

Out of the kernel of this operator, and of its inverse, hierarchies of integrable evolution equations arise which are symmetries of each other[?] (this will be worked out in detail for the KdV case later on).

According to[?], given a hereditary operator R and a flow vector field (labelled with a number) V_1 such that $\mathcal{L}_{V_1} R = 0$, i.e., R is a **recursion** operator for V_1 , then a **hierarchy** is defined as a semi-infinite collection of evolution vectors: $\{V_j = R^{j-1} \cdot V_1, \quad j = 1, \dots, \infty\}$,

which are symmetries of each other: $\mathcal{L}_{V_i} V_j = 0$, $i, j \geq 1$, and thus every evolution vector in the hierarchy defines an evolution equation which is integrable. In the KdV hierarchy, the KdV equation is the second member (V_2). The first vector (V_1) represents the translation symmetry, and it is shown to generate the kernel of the inverse hereditary operator, R^{-1} . By convention, we refer to the above as a **positive hierarchy**.

The hereditary property for the operator R may be used to show formally that R^{-1} is also hereditary.[?] Therefore, we could conjecture that new hierarchies (referred to as **negative hierarchies**) of evolution vectors may be constructed, which first members generate the kernel of the operator R , and successive members are defined by contraction of the first members with powers of the operator R^{-1} . In the KdV case, there are three negative hierarchies. These new negative equations include the ShG, Liouville, Camassa–Holm, and the Hunter–Zheng equations.

F. Notation: the Positive and Negative Hierarchies in terms of the Hereditary Operators

The analysis is restricted to the KdV hierarchies, but it is easily generalizable to other systems related to hereditary operators.

We will adopt the following notation for evolution vectors in the hierarchies:

$$u_t = V_n^{(k)}[u],$$

where $k = 1$ denotes the positive hierarchy; $k = -1, -2, -3$ for the three negative hierarchies, and $n = 1, \dots, \infty$ denotes the place of a vector within the hierarchy, so that we have $n = 1$ for the first vector of each hierarchy, i.e., the relevant generator of the kernel of $R^{-\text{sgn}(k)}$.

$$\begin{aligned} R^{-1}[u] \cdot V_1^{(1)}[u] &= 0, \\ R[u] \cdot V_1^{(-1)}[u] &= R[u] \cdot V_1^{(-2)}[u] = R[u] \cdot V_1^{(-3)}[u] = 0. \end{aligned} \tag{8}$$

Successive members in the hierarchies are defined by recurrence:

$$V_{n+1}^{(k)}[u] = (R[u])^{\text{sgn}(k)} \cdot V_n^{(k)}[u], \quad n \geq 1, \quad k = 1, -1, -2, -3.$$

In this way, the positive hierarchy begins with the vector $V_1^{(1)}[u] = -u_x$, continues with the KdV vector $V_2^{(1)}[u] = -u_{xxx} - 12u u_x$, and so on (these vectors were called V_1 and V_2 in the last subsection).

For the negative hierarchy, as the operator R^{-1} is harder to work with, there is a recurrent way of writing the negative vectors, in terms of R :

$$V_n^{(k)}[u] = R[u] \cdot V_{n+1}^{(k)}[u], \quad n \geq 1, \quad k = -1, -2, -3. \quad (9)$$

The explicit expression for negative vectors relies on the factorized form^{?,?} of the hereditary operator, and will be realized in section IV in terms of nonlocal fields which, however, are tractable in the same scheme as the local ones.

III. LADDERS OF ACTION PRINCIPLES AND CONSTANTS OF THE MOTION

Complementary to the well-known bi-Hamiltonian formulation,[?] we may find a bi-symplectic or multi-symplectic structure starting from the hereditary property. Assume that we have a Nijenhuis operator R along with one closed 2-form $\Sigma^{(1)}$ such that $\Sigma^{(2)} \equiv \Sigma^{(1)} \cdot R$ be a closed 2-form: then, the two semi-infinite dimensional sets (**symplectic ladders**) of 2-forms

$$\{\Sigma^{(n)} \equiv \Sigma^{(1)} \cdot R^{n-1}, \quad n = 1, \dots, \infty\} \quad (\text{positive symplectic ladder}),$$

and

$$\{\Sigma^{(n)} \equiv \Sigma^{(1)} \cdot R^{n-1}, \quad n = 0, -1, \dots, -\infty\} \quad (\text{negative symplectic ladder}),$$

contain only closed 2-forms. The distinction between positive and negative ladders is somewhat arbitrary, for it depends on which hereditary operator, R or R^{-1} , is being used, and which symplectic 2-form is taken as $\Sigma^{(1)}$.

The proof of the above statement is very simple. In fact, it is equivalent to the proof for the so-called Poisson pencil or set of compatible implectic operators[?] for multi-Hamiltonian theories, after defining the implectic or Hamiltonian operators as the inverses of the 2-forms in the ladder.

The above result is independent of any evolution vector. When we consider the vectors in the hierarchies, however, it is easily checked (as it holds in the examples) that $\mathcal{L}_{V_1} \Sigma^{(1)} = 0$. Therefore, using Leibnitz rule, all the 2-forms in the ladder are symplectic operators for the first evolution vector in the hierarchy.

Using the identity $\mathcal{L}_{R \cdot \eta} \Sigma - \mathcal{L}_\eta (\Sigma \cdot R) = i_{R \cdot \eta} \delta \Sigma - i_\eta \delta (\Sigma \cdot R)$, which holds for any vector η , $(1, 1)$ tensor R and 2-form Σ , where i_η stands for interior product (contraction of the vector

η with the left component of a p -form), we obtain the important result for any hierarchy:

$$\mathcal{L}_{V_j} \Sigma^{(n)} = 0, \quad j = 1, 2, \dots, \infty, \quad n = -\infty, \dots, \infty, \quad (10)$$

which means that a ladder of action principles may be constructed for every evolution vector in the hierarchy (in particular, for the KdV equation).

This fact is used in⁷ to construct action principles, with the only drawback it needs to integrate the 2-forms (Poincaré lemma) in order to get the action principles.

Let us assume for the rest of this section that we have a Nijenhuis operator R along with its inverse R^{-1} , a symplectic ladder $\{\Sigma^{(n)}, \quad n = -\infty, \dots, \infty\}$, and a hierarchy $\{V_j, \quad j = 1, \dots, \infty\}$ with the corresponding properties mentioned above.

The purpose of the following subsection is to construct, for the second evolution vector in the positive hierarchy (though the analysis is easily extended for other positive and negative evolution vectors), the action principles associated to each of the above symplectic 2-forms. These action principles are involved in the explicit construction of constants of the motion for the evolution equation.

A. Construction of Action Principles and Constants of the Motion out of Symmetries and Symplectic Operators

Heuristically, if we had a symmetry for a given evolution equation we could obtain in a direct way (by contraction of it with any symplectic 2-form) a Lagrangian 1-form and therefore an action principle for that equation.

We have done this procedure for any equation in the positive hierarchy, using the Galilean symmetry, obtaining as a result a ladder of action principles which are (explicitly) time-dependent (in fact, linear in time). For simplicity, however, we rewrite the actions as time-independent objects, and the discussion will be restricted to the second vector (which corresponds to the KdV equation), which is from now on referred to as the vector V_2 .

The following definition will be a key to the construction of the ladder of action principles for the evolution vector V_2 , and it permits a generalization to the negative hierarchies as well as to other systems (e.g., nonlinear Schrödinger equation):

Definition 5. The Galilean vector field η_{gal} is a time-independent vector field, defined by

three properties:

$$\begin{aligned}\mathcal{L}_{\eta_{\text{gal}}} R &= \mathbb{I}, \\ \mathcal{L}_{\eta_{\text{gal}}} \Sigma^{(1)} &= 0, \\ \mathcal{L}_{\eta_{\text{gal}}} V_2 &= \alpha V_1,\end{aligned}\tag{11}$$

where α is a numeric constant.

As a consequence of the definition, it turns out that η_{gal} is a Mastersymmetry[?] for the hierarchy $\{V_j, \ j = 1, \dots, \infty\}$. Explicitly, we have $\mathcal{L}_{V_{j+1}} \eta_{\text{gal}} = (\alpha + j - 1) V_j$, for $j = 1, \dots, \infty$.

The aim is to construct time-independent standard Lagrangian pairs $(P; K)$, where $\mathcal{L}_{V_2} P = \delta K$. The action principles will read $S[q^a(t)] = \int_{t_-}^{t_+} dt (P_a(\dot{q}^a - V_2^a) + K)$, and the Euler-Lagrange equations will involve the symplectic 2-forms in the ladder, i.e., $\delta P = \Sigma$.

Theorem 6. *The 1-forms defined by $P^{(m)} \equiv i_{\eta_{\text{gal}}} \Sigma^{(m+1)}$, for $m = -\infty, \dots, \infty$ are “integrals” of the symplectic 2-forms in the ladder. That is to say,*

$$\delta P^{(m)} = m \Sigma^{(m)}, \quad m = -\infty, \dots, \infty.\tag{12}$$

Proof. If we take the exterior derivatives of the 1-forms, using the identity

$$\mathcal{L}_{\eta} = i_{\eta} \delta + \delta i_{\eta},\tag{13}$$

which holds (for every vector η) when operating on any p -form, we find

$$\delta P^{(m)} = \mathcal{L}_{\eta_{\text{gal}}} \Sigma^{(m+1)} = \mathcal{L}_{\eta_{\text{gal}}} (\Sigma^{(1)} \cdot R^m) = \Sigma^{(1)} \cdot \mathcal{L}_{\eta_{\text{gal}}} (R^m) = m \Sigma^{(1)} \cdot R^{m-1} = m \Sigma^{(m)}, m \in \mathbb{Z}$$

after using the definition of Galilean vector and Leibnitz rule. \square

In order to complete the action principles, there remains to find the second members of the corresponding standard Lagrangian pairs.

Theorem 7. *For each $m \in \mathbb{Z}$, the pair $(P^{(m)}; K^{(m)})$ with $K^{(m)} \equiv \frac{\alpha}{m+\alpha} i_{V_2} P^{(m)}$, is a standard Lagrangian pair for the evolution equation $\dot{q}^a = V_2^a$, i.e., $\mathcal{L}_{V_2} P^{(m)} = \delta K^{(m)}$. For $m \neq 0$, the action principle is*

$$S^{(m)}[q^a(t)] = \int_{t_-}^{t_+} P^{(m)}_a \left(\dot{q}^a - \frac{m}{m+\alpha} V_2^a \right) dt.\tag{14}$$

Moreover, the 0-forms $K^{(m)}$ are constants of the motion for the evolution equation, for $m \in \mathbb{Z}$:

$$\mathcal{L}_{V_2} K^{(m)} = 0.$$

The above is a Noetherian way to construct constants of the motion, for all the action principles in the ladder for KdV are naturally invariant under the KdV flow V_2 itself (see equation (10)).

Remark 8. The case $m = 0$ would lead to a trivial action principle from equation (14), for the Euler–Lagrange equations are identically zero: it is shown that this case leads to a time-dependent constant of the motion. This does not mean that the symplectic 2-form $\Sigma^{(0)}$ defines a trivial action principle. In fact, its associated action principle may be found by hand (see the end of this subsection), and it is related to the usual action principle for the ShG equation (see subsection VB).

Proof: Lagrangian Pairs. For $m \in \mathbb{Z}$, take Lie derivatives of the 1-forms $P^{(m)}$ along the evolution vector V_2 , using Leibnitz rule:

$$\mathcal{L}_{V_2} P^{(m)} = \mathcal{L}_{V_2} i_{\eta_{\text{gal}}} \Sigma^{(m+1)} = -\alpha i_{V_1} \Sigma^{(m+1)} = -\alpha i_{V_2} \Sigma^{(m)}. \quad (15)$$

But, using the identity (13) and the result (12) we rewrite the last expression to get

$$m \mathcal{L}_{V_2} P^{(m)} = -\alpha \left(\mathcal{L}_{V_2} P^{(m)} - \delta i_{V_2} P^{(m)} \right),$$

therefore

$$\mathcal{L}_{V_2} P^{(m)} = \delta \left(\frac{\alpha}{m + \alpha} i_{V_2} P^{(m)} \right) \equiv \delta K^{(m)}. \quad (16)$$

□

Proof: Constants of the Motion. We use the above result (16), to find

$$\mathcal{L}_{V_2} K^{(m)} = \frac{\alpha}{m + \alpha} i_{V_2} \mathcal{L}_{V_2} P^{(m)} = \frac{\alpha}{m + \alpha} i_{V_2} \delta K^{(m)} = \frac{\alpha}{m + \alpha} \mathcal{L}_{V_2} K^{(m)},$$

which implies $\mathcal{L}_{V_2} K^{(m)} = 0$, $m \neq 0$.

For $m = 0$ we find a weaker result: Equation (15) implies $\delta \mathcal{L}_{V_2} K^{(0)} = 0$. Therefore $\mathcal{L}_{V_2} K^{(0)} = c$, is a number (usually equal to zero) that may be absorbed to define a time-dependent constant of the motion: $\tilde{K}^{(0)}(t) = K^{(0)} - ct$. □

For the KdV equation, when $m \geq -1$ we get the usual denumerably infinite set of constants of the motion.[?] Notice that this theorem represents also a constructive method to obtain such constants. On the other hand, when $m \leq -2$ the constants are numerical or vanishing boundary terms. Amazingly, this fact allows one to construct an infinite number of nonlocal constants of the motion for KdV, using the nonlocal action principles (see subsection III C).

Proof: Action Principles. The action principles (14) arise directly from equation (3), using the definition of $K^{(m)}$. \square

From the point of view of theorem 7, the case $m = 0$ also leads to a time-dependent constant of the motion. From equation (12), it follows that $P^{(0)} = \delta C^{(0)}$, and thus $\mathcal{L}_{V_2} C^{(0)} = K^{(0)}$. But we know that $\mathcal{L}_{V_2} K^{(0)} = c$ is a number. We obtain the following time-dependent constant of the motion for the evolution vector V_2 :

$$C[q^a(t), t] = C^{(0)} - t K^{(0)} + \frac{c t^2}{2}. \quad (17)$$

Finally, for the case $m = 0$, a special (“missing”) action principle is constructed by hand from integration of the 2-form $\Sigma^{(0)}$, which leads to the 1-form $P^{(M)}$, such that $\delta P^{(M)} = \Sigma^{(0)}$. We will have $\mathcal{L}_{V_2} P^{(M)} = \delta K^{(M)}$, and the action is

$$S^{(0)}[q^a(t)] = \int_{t_-}^{t_+} (P^{(M)}_a (\dot{q}^a - V_2^a) + K^{(M)}) dt.$$

B. The Euler–Lagrange Equations as Deformed Evolution Equations

The Euler–Lagrange equations that arise from variation of each action $S^{(m)}$, $m \in \mathbb{Z}$ are, apart from nonzero numeric factors,

$$\Sigma^{(m)}_{ab}(\dot{q}^b - V_2^b) = 0, \quad m \in \mathbb{Z}.$$

The kernel of the symplectic operators, $\text{Ker} \Sigma^{(m)}$, is of importance here. For each action principle we obtain an equivalent, **deformed**, evolution equation:

$$\dot{q}^a = V_2^a + \sum_{j=1}^{N_m} \theta_j \eta_{j;m}^a,$$

where $N_m = \dim(\text{Ker} \Sigma^{(m)})$, the vectors $\{\eta_{j;m}\}_{j=1}^{N_m}$ generate the kernel of $\Sigma^{(m)}$, and $\theta_j = \theta_j(t)$ are arbitrary 0-forms: it can be said that these Euler–Lagrange equations and the

action principles acquire extra symmetries (as compared to the symmetries of the original equations).

As the 2-forms here are formed by contraction of powers of the hereditary operators R and R^{-1} with $\Sigma^{(1)}$, it is clear that the kernel of the 2-forms are computed essentially from vectors in the kernel of the operators R^m and R^{-m} , for $m > 0$: as we have mentioned, these are the positive and negative evolution vectors. In section IV, we will find explicitly the deformed equations for the KdV equation in terms of the positive and negative vectors.

C. Construction of nonlocal Constants of the Motion from Symmetries and non-standard Lagrangian 1-forms

Let us assume, as it will be demonstrated in subsection IV H for the KdV case (under usual boundary conditions), that the constants of the motion from theorem 7 are $K^{(m)} = 0$ or a numeric constant for $m \leq -2$. This implies that $\mathcal{L}_{V_2} P^{(m)} = 0$, i.e., $P^{(m)}$ is a non-standard Lagrangian 1-form for the flow V_2 . Assume also that the evolution equation defined by the flow V_2 possesses a symmetry η . Then

Theorem 9. *The 0-forms defined by $Q^{(m)} \equiv i_\eta P^{(m)}$, for $m \leq -2$, are constants of the motion for the flow V_2 , i.e., $\left(\partial_t + \mathcal{L}_{V_2}\right) Q^{(m)} = 0$ for $m \leq -2$.*

Proof. The proof follows directly from Leibnitz rule. □

The above is a non-Noetherian way[?] to construct constants of the motion, in the sense that the action principles need not be invariant under the relevant symmetry. In subsection IV J we construct “generating functions” for three infinite-dimensional sets of nonlocal constants of the motion for the KdV equation, setting η as a nonlocal internal symmetry for KdV.

IV. EXAMPLE: THE KDV EQUATION

A. Known Objects

We begin by presenting the Nijenhuis operator relevant for the KdV hierarchy: as an operator,[?] $R[u] = D^2 + 8u + 4u_x D^{-1}$, where D and D^{-1} are, respectively, the derivative and the anti-derivative operators: $Df(x) \equiv \frac{\partial f(x)}{\partial x}$, $D^{-1}g(x) \equiv \int_{x_-}^{x_+} \epsilon(x - x') g(x') dx'$, with $\epsilon(x - x') = 1/2 \text{sign}(x - x')$.

Next, the **positive** hierarchy begins with the vector $V_1^{(1)}[u] = -u_x$. The second vector in the positive hierarchy is obtained after application of the Nijenhuis operator on the latter vector: $V_2^{(1)}[u] = R[u] \cdot V_1^{(1)}[u] = -u_{xxx} - 12u u_x \equiv V_2[u]$. We see it represents the KdV equation (2).

Next, we write the first symplectic 2-form: as an operator,[?] $\Sigma^{(1)}[u] = D^{-1}$. The second symplectic operator is constructed just by contracting the latter operator with the Nijenhuis operator: $\Sigma^{(2)}[u] = \Sigma^{(1)}[u] \cdot R[u] = D + 4u D^{-1} + 4D^{-1}u$. These operators are closed under usual boundary conditions for the vector fields: the ladder, then, contains only closed 2-forms. Now, it is easy to show that these operators are symplectic for the flow defined by $V_1^{(1)}$, therefore all operators in the ladder are symplectic for the KdV flow $V_2^{(1)}$, as it is stated in equation (10).

Finally, the Galilean vector is just the time-independent part of the Galilean symmetry (7): $\eta_{\text{gal}}[u] = \frac{1}{8}$, and the constant in the last of the defining equations (11) is $\alpha = 3/2$.

B. Explicit form of the KdV Negative Hierarchies: Linear generalization and factorization of the Hereditary Operator

In order to find explicitly the KdV **negative** hierarchies, we factorize a generalization of the hereditary operator R , which is obtained by addition of a multiple of the identity tensor \mathbb{I} :

$$R(\lambda)[u] \equiv R[u] + 4\lambda \mathbb{I}.$$

This is also a hereditary operator, for fixed λ , which is taken as an arbitrary real number.

The idea behind this generalization, is that the kernel of $R(\lambda)[u]$ contains all the negative hierarchies in its Taylor expansion around $\lambda = 0$, so we will write the negative hierarchies in a compact way.

Lemma 10. *The vectors $V_1^{(-1)}(\lambda)[u]$, $V_1^{(-2)}(\lambda)[u]$, $V_1^{(-3)}(\lambda)[u]$, defined by*

$$V_1^{(k)}(\lambda)[u] \equiv \sum_{n=0}^{\infty} (-4\lambda)^n V_{n+1}^{(k)}[u], \quad k = -1, -2, -3$$

generate the vectorial kernel of $R(\lambda)[u]$.

Conversely, all vectors in the negative hierarchies may be obtained from the vectorial kernel of $R(\lambda)[u]$:

$$V_{n+1}^{(k)}[u] = \frac{1}{n!} (-4)^{-n} \left. \frac{\partial^n}{\partial \lambda^n} V_1^{(k)}(\lambda)[u] \right|_{\lambda=0}, \quad k = -1, -2, -3, \quad n = 0, 1, \dots, \infty. \quad (18)$$

Proof. Consider the action of $R[u]$ on the vector $V_1^{(k)}(\lambda)[u]$. Using equations (8) and (9), we get

$$R[u] \cdot V_1^{(k)}(\lambda)[u] = \sum_{n=0}^{\infty} (-4\lambda)^n R[u] \cdot V_{n+1}^{(k)}[u] = \sum_{n=1}^{\infty} (-4\lambda)^n V_n^{(k)}[u] = -4\lambda V_1^{(k)}(\lambda)[u],$$

therefore $R(\lambda)[u] \cdot V_1^{(k)}(\lambda)[u] = (R(\lambda)[u] + 4\lambda) \cdot V_1^{(k)}(\lambda)[u] = 0$. \square

The factorization process[?] implies the definition of auxiliary fields, which are directly related to nonlocal pre-potentials found in the literature[?] and to the associated isospectral linear eigenvalue problem:[?]

$$\psi_{xx} + 2u\psi = -\lambda\psi, \quad (19)$$

where $\psi = \psi(x, t; \lambda)$. As usual, we assume $\lambda_t = 0$, and $u = u(x, t)$ is independent of λ .

Alternatively, we write the above equation as: $L(\lambda) \cdot \psi = 0$, where $L(\lambda) = D^2 + 2u + \lambda$ or, in a factorized way, $L(\lambda) = \frac{1}{\psi} D \psi^2 D \frac{1}{\psi}$, is the **Lax operator**. The elements in the kernel of this operator are solutions of the linear problem (19). Two l.i. solutions are: $\psi(\lambda) \equiv \psi(x, t; \lambda)$, and

$$\bar{\psi}(\lambda) \equiv \psi(\lambda) D^{-1} (1/\psi(\lambda)^2). \quad (20)$$

The above eigenvalue problem may be understood as an extended coordinate system labelled by $\psi(x, t; \lambda)$, with λ as an additional variable (just like x), and which however must solve an extra equation, $L(\lambda) \cdot \psi(\lambda) = 0$, which we call **constraint**. This constraint lets us write derivatives of the field $\psi(\lambda)$ with respect to λ in terms of the field itself, in a nonlocal way. This will be useful in the next section, when we write the negative ladder of action principles. We obtain, apart from integration constants,

$$\begin{aligned} \frac{\partial^n}{\partial \lambda^n} \psi &= (-1)^n n! L(\lambda)^{-n} \psi, \\ \frac{\partial^n}{\partial \lambda^n} \bar{\psi} &= (-1)^n n! L(\lambda)^{-n} \bar{\psi}, \quad n \geq 1, \end{aligned} \quad (21)$$

where $L(\lambda)^{-1} = \psi D^{-1} \frac{1}{\psi^2} D^{-1} \psi$, $\psi = \psi(\lambda)$ and $\bar{\psi} = \bar{\psi}(\lambda)$.

Now, the factorization of $R(\lambda)[u]$ is found to be

$$R(\lambda)[u] = \frac{1}{\psi(\lambda)^2} D \psi(\lambda)^2 D \psi(\lambda)^2 D \frac{1}{\psi(\lambda)^2} D^{-1}.$$

It is remarkable that the above operator is *linear* in λ , which is a consequence of the constraint (19).

The kernel of the operator $R(\lambda)[u]$ is easily found to be composed by three nonlocal vectors:

$$\begin{aligned} u_t &= V_1^{(-1)}(\lambda)[u] = (\psi(\lambda)^2)_x, \\ u_t &= V_1^{(-2)}(\lambda)[u] = (\psi(\lambda) \bar{\psi}(\lambda))_x, \\ u_t &= V_1^{(-3)}(\lambda)[u] = (\bar{\psi}(\lambda)^2)_x, \end{aligned} \tag{22}$$

where $\psi(\lambda)$, $\bar{\psi}(\lambda)$ (see equation (20)) are two l.i. solutions of the constraint (19). These vectors contain the whole negative hierarchies if λ is left arbitrary, as lemma 10 states. A discrete infinite-dimensional representation of these negative vectors, as nonlocal symmetries of the KdV equation, is known.⁷ On the other hand, the continuous representation (22) of these vectors, more suitable for field theory, allows for considering them also as possible evolution equations, which will be recognized as known integrable equations in subsection V A.

C. Explicit form and Kernel of the Inverse Hereditary Operator R^{-1}

The inverse hereditary operator, $R^{-1}(\lambda)$, is found easily after inverting every factor. Assuming appropriate boundary conditions on the field ψ , we get:

$$R^{-1}(\lambda)[u] = D \psi(\lambda)^2 D^{-1} \frac{1}{\psi(\lambda)^2} D^{-1} \frac{1}{\psi(\lambda)^2} D^{-1} \psi(\lambda)^2.$$

It is easy to show that the kernel of this operator is generated by the vector $V_1^{(1)} = -u_x$.

D. Factorization of Positive and Negative Symplectic and Hamiltonian Operators

Using the factorized form of the hereditary operators, we get easily the symplectic operators in factorized form. For the positive ones, we have

$$\begin{aligned} \Sigma^{(2)}[u] &= D^{-1} \frac{1}{\psi^2} D \psi^2 D \psi^2 D \frac{1}{\psi^2} D^{-1}, \\ \Sigma^{(3)}[u] &= D^{-1} \frac{1}{\psi^2} D \psi^2 D \psi^2 D \frac{1}{\psi^2} D^{-1} \frac{1}{\psi^2} D \psi^2 D \psi^2 D \frac{1}{\psi^2} D^{-1}, \end{aligned}$$

and so on, where $\psi = \psi(\lambda = 0)$. Notice that the inverses of these operators give new nonlocal Hamiltonian operators for KdV.

For the negative ones, on the other hand, we have

$$\begin{aligned} \Sigma^{(0)}[u] &= \psi^2 D^{-1} \frac{1}{\psi^2} D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2, \\ \Sigma^{(-1)}[u] &= \psi^2 D^{-1} \frac{1}{\psi^2} D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2 D \psi^2 D^{-1} \frac{1}{\psi^2} D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2, \end{aligned} \tag{23}$$

and so on. By the way, the above expression for $\Sigma^{(0)}[u]$ turns out to solve a puzzle in the recent literature,⁷ for it is the inverse of Magri's Hamiltonian operator. As we see, we have got all inverses (in factorized form) of all Hamiltonian operators within the multi-Hamiltonian structure for KdV.

There is another, concise way to write these negative operators, which resembles the way we wrote the negative vectors in terms of the λ -dependent first one. We state the lemma without proof:

Lemma 11. *The negative symplectic operators for KdV are written in terms of $\Sigma^{(0)}(\lambda)[u] \equiv \Sigma^{(1)} \cdot R^{-1}(\lambda)[u]$ in the following way:*

$$\Sigma^{(-n)}[u] = \frac{1}{n!} (-4)^{-n} \frac{\partial^n}{\partial \lambda^n} \Sigma^{(0)}(\lambda)[u] \Big|_{\lambda=0}, \quad n \geq 1.$$

A similar formula may be written for the nonlocal Hamiltonian operators.

E. Negative Vectors as Kernel of Positive Symplectic Operators

The kernel spaces $\text{Ker} \Sigma^{(n)}[u]$, for $n = 1, \dots, \infty$, are easily computed in terms of the kernel of positive powers of R , from the fact that $\text{Ker} \Sigma^{(1)}[u]$ is null. We get

$$\text{Ker} \Sigma^{(n+1)} = \text{span}\{V_m^{(k)}[u], \quad k = -1, -2, -3, \quad m = 1, \dots, n\}, \quad n \geq 0.$$

F. Positive Vectors as Kernel of Negative Symplectic Operators

Finally we compute the kernel spaces $\text{Ker} \Sigma^{(n)}[u]$, for $n = 0, -1, \dots, -\infty$. It is easily seen that the operator $\Sigma^{(0)}[u]$ has a null kernel. This time we have to evaluate the kernel of negative powers of R . We get

$$\text{Ker} \Sigma^{(-n)}[u] = \text{span}\{V_m^{(1)}[u], \quad m = 1, \dots, n\}, \quad n > 0,$$

so that, in particular, the action principle associated to $\Sigma^{(-2)}[u]$ for the KdV equation has the translation vector as well as the KdV vector as generators of its kernel, therefore the action has to be time-reparametrization invariant⁷ (we will see an example soon).

G. Action Principles for KdV: Positive Lagrangian Ladders

Remark 12. If $P_a \delta u^a$ denotes a 1-form, where $a \equiv x$ is a continuous index, we will write the component P_a of the 1-form as $\mathcal{P}(x, t)$ (which looks more like a density) when dealing with it inside an integral sign.

Following theorems 6 and 7, we write the action principles from equation (14):

$$S^{(m)}[u(x, t)] = \int_{t_-}^{t_+} \int_{x_-}^{x_+} \mathcal{P}^{(m)}[u](x, t) \left(u_t + \frac{m}{m+3/2} (u_{xxx} + 12 u u_x) \right) dx dt, \quad (24)$$

for $m > 0$, where

$$\mathcal{P}^{(1)}[u](x, t) = i_{\eta_{\text{gal}}} \Sigma^{(2)}[u] = -(D + 4 u D^{-1} + 4 D^{-1} u) \frac{1}{8} = -\frac{1}{2} (x u + D^{-1}(u)) ,$$

and successive 1-forms are defined by recurrence: $\mathcal{P}^{(m+1)}[u](x, t) = R^\dagger[u] \cdot \mathcal{P}^{(m)}[u](x, t)$, where $R^\dagger[u]$ is the transpose Nijenhuis operator.

The action principle $S^{(1)}[u(x, t)]$ gives rise to the following Euler–Lagrange equations:

$$D^{-1}(u_t + u_{xxx} + 12 u u_x) = 0 ,$$

which are equivalent to KdV. The associated constant of the motion is:

$$H^{(1)}[u] \equiv i_{V_2} P^{(1)} = \frac{1}{2} \int dx (x u + D^{-1}(u)) (u_{xxx} + 12 u u_x) = \frac{5}{4} \int dx (u_x^2 - 4 u^3) ,$$

which is a member of the known set.

The next action principle is written as equation (24), with

$$\mathcal{P}^{(2)}[u](x, t) = R^\dagger[u] \cdot \mathcal{P}^{(1)}[u](x, t) = -\frac{1}{2} (3 u_x + x u_{xx} + 6 x u^2 + 4 u D^{-1}(u) + 6 D^{-1}(u^2)) .$$

The Euler–Lagrange equations are: $2 (D + 4 u D^{-1} + 4 D^{-1} u) (u_t + u_{xxx} + 12 u u_x) = 0$ or, in factorized form, $2 D^{-1} \frac{1}{\psi^2} D \psi^2 D \psi^2 D \frac{1}{\psi^2} D^{-1} (u_t + u_{xxx} + 12 u u_x) = 0$. These equations are not equivalent to the original equations: instead, they are equivalent to the deformed equations

$$u_t = -u_{xxx} - 12 u u_x + \theta_1 (\psi^2)_x + \theta_2 (\psi \bar{\psi})_x + \theta_3 (\bar{\psi}^2)_x ,$$

where θ_j are arbitrary 0-forms that multiply the generators of $\text{Ker} \Sigma^{(2)}[u]$, and $\psi = \psi(\lambda = 0)$.

We could continue this process **constructively**, obtaining explicitly the constants of the motion and the action principles as well as the Euler–Lagrange equations: the original KdV equation gets deformed with vectors in the negative hierarchies, as it was mentioned before.

H. Action Principles for KdV: Negative Lagrangian Ladders

Now we turn to the construction of the negative action principles for KdV, whose action functionals are defined by

$$S^{(m)}[u(x, t)] = \int_{t_-}^{t_+} \int_{x_-}^{x_+} \mathcal{P}^{(m)}[u](x, t) \left(u_t + \frac{m}{m+3/2} (u_{xxx} + 12 u u_x) \right) dx dt,$$

for $m < 0$. We only need to evaluate $\mathcal{P}^{(-m)}[u](x, t) = (R^\dagger[u])^{-m} \cdot \mathcal{P}^{(0)}[u](x, t)$, for $m < 0$. This is done easily after stating the following corollary (from lemma 11):

Corollary 13. *The negative 1-forms are obtained from the first negative one as follows:*

$$\mathcal{P}^{(-n-1)}[u](x, t) = \frac{1}{n!} (-4)^{-n} \frac{\partial^n}{\partial \lambda^n} \mathcal{P}^{(-1)}(\lambda)[u](x, t) \Big|_{\lambda=0}, \quad n \geq 1,$$

where $\mathcal{P}^{(-1)}(\lambda)[u](x, t) = -\frac{1}{8} \psi^2 D^{-1} \frac{1}{\psi^2} D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2$ or, using equation (21),

$$\mathcal{P}^{(-1)}(\lambda)[u](x, t) = \frac{1}{16} (\psi_\lambda \bar{\psi} - \psi \bar{\psi}_\lambda).$$

The first negative action principle from equation (24) is thus:

$$S^{(-1)}[u(x, t)] = \int_{t_-}^{t_+} \int_{x_-}^{x_+} \frac{1}{16} (\psi_\lambda \bar{\psi} - \psi \bar{\psi}_\lambda) \left(u_t + \frac{-1}{-1+3/2} (u_{xxx} + 12 u u_x) \right) dx dt,$$

or, after some manipulations, $S^{(-1)}[u(x, t)] = \int_{t_-}^{t_+} \int_{x_-}^{x_+} \left(\frac{1}{16} (\psi_\lambda \bar{\psi} - \psi \bar{\psi}_\lambda) u_t - \frac{1}{4} u \right) dx dt$, where we have to evaluate the fields $\psi, \bar{\psi}$ at $\lambda = 0$. This action principle is highly nonlocal, even in terms of the auxiliary fields (see equation (21)). However, the Euler–Lagrange equations are obtained as usual, varying the action with respect to the field u , and using the appropriate transformation matrices. We obtain $-\Sigma^{(-1)}[u] \cdot (u_t + u_{xxx} + 12 u u_x) = 0$, or explicitly, using the fact that the kernel of this operator is generated by $V_1^{(1)}$,

$$u_t = -u_{xxx} - 12 u u_x + \theta_1 u_x,$$

where θ_1 is arbitrary.

We stress there is no need to hesitate about the inclusion of auxiliary fields in the negative action principles, for they are not varied independently. Alternatively, we may map the above action into a mixed action principle, in which the fields ψ, u and a Lagrange multiplier ρ are varied independently:

$$S^{(-1)}[\psi(x, t), u(x, t), \rho(x, t)] = \int_{t_-}^{t_+} \int_{x_-}^{x_+} \left(\frac{1}{16} \frac{\psi_t}{\psi^3} (D^{-1} \psi^2) + \frac{1}{8} \frac{\psi_{xx}}{\psi} + \rho \left(u + \frac{\psi_{xx}}{2\psi} \right) \right) dx dt. \quad (25)$$

See⁷ for a general discussion.

The next negative action principles are quite simple. Recall that the evolution vector itself V_2 is in the kernel of the symplectic operators $\Sigma^{(m)}$, for $m = -2, \dots, -\infty$, so that the action principles should be time-reparametrization invariant.⁷ From equation (14), the only chance is $K^{(m)} \propto i_{V_2} P^{(m)} = 0$ or a numeric constant (which would not change the action principle), for $m \leq -2$. This is easily shown, from the fact that the interior product $I(\lambda) = i_{V^{(1)}} P^{(-1)}(\lambda)[u] = -\frac{1}{16} \int_{x_-}^{x_+} dx (\psi_\lambda \bar{\psi} - \psi \bar{\psi}_\lambda) u_x$, where $\psi = \psi(\lambda)$, is a numeric constant for all λ : $I(\lambda) = -\frac{1}{32}(x_+ - x_-)$. We get, then, manifestly time-reparametrization invariant actions:

$$S^{(-n-1)}[u(x, t)] = \frac{1}{16 (n!)} (-4)^{-n} \int_{t_-}^{t_+} \int_{x_-}^{x_+} \frac{\partial^n}{\partial \lambda^n} (\psi_\lambda \bar{\psi} - \psi \bar{\psi}_\lambda) \Big|_{\lambda=0} u_t dx dt, \quad n \geq 1,$$

and we may use equation (21) in order to write the λ -derivatives in terms of nonlocal expressions. For example, the second negative action principle is

$$S^{(-2)}[u(x, t)] = -\frac{1}{64} \int_{t_-}^{t_+} \int_{x_-}^{x_+} (\psi_{\lambda\lambda} \bar{\psi} - \psi \bar{\psi}_{\lambda\lambda}) \Big|_{\lambda=0} u_t dx dt,$$

where $\psi_{\lambda\lambda} = 2 \psi D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2 D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2$ and $\bar{\psi} = \psi D^{-1}(1/\psi^2)$; the Euler-Lagrange equations are equivalent to

$$u_t = \theta_1 (u_{xxx} + 12 u u_x) + \theta_2 u_x$$

where, as usual, θ_j are arbitrary functionals. The invariance $t \rightarrow \tau(t)$ is evident.

I. The Missing Action Principle for KdV, a time-dependent Constant of the Motion and the Internal Vectors

So far we have obtained two ladders of action principles for the KdV equation: the positive (quasi local) and the negative ladders (highly nonlocal). However, there is a missing action principle: this is the case $m = 0$, which is actually twofold: first, the 1-form $P^{(0)}[u] \equiv i_{\eta_{\text{gal}}} \Sigma^{(1)}[u] = -x/8$ is closed: $P^{(0)}[u] = \delta C^{(0)}[u]$, where $C^{(0)}[u] = -\int_{x_-}^{x_+} dx x u/8$. From equation (17), we obtain a known⁷ time-dependent constant of the motion for KdV: $C[u, t] = \frac{1}{8} \int_{x_-}^{x_+} (6 t u^2 - x u) dx$.

Second, the action principle for the symplectic 2-form $\Sigma^{(0)}[u]$ (see equation (23)) has to be evaluated by hand. After some hard but straightforward calculations, we find that the 1-form $\mathcal{P}^{(M)}[u](x, t) \equiv \frac{\psi^2}{4} D^{-1} \left(\frac{1}{\psi^2} \ln \psi \right)$ is a solution of $\delta P^{(M)} = \Sigma^{(0)}$.

In this case, we map to the ψ -coordinate system for simplicity. We get the action principle

$$S^{(0)}[\psi, u, \rho] = -\frac{1}{8} \int_{x_-}^{x_+} \left(\frac{\psi_x \psi_t}{\psi^2} - \frac{\psi_{xx}^2}{\psi^2} + \rho \left(u + \frac{\psi_{xx}}{2\psi} \right) \right) dx dt.$$

The Euler–Lagrange equations we obtain are: for the field ψ ,

$$\frac{1}{\psi} D \frac{1}{\psi} \left(\psi_t + \psi_{xxx} - 3 \frac{\psi_x \psi_{xx}}{\psi} \right) = 0,$$

for the field u : $u = -\frac{\psi_{xx}}{2\psi} \Rightarrow u_t = -u_{xxx} - 12 u u_x$, and for the Lagrange multiplier: $\rho = 0$.

Notice that the Euler–Lagrange equations for ψ are equivalent to

$$\psi_t = -\psi_{xxx} + 3 \frac{\psi_x \psi_{xx}}{\psi} + \theta_1 \psi,$$

where θ_1 is arbitrary. This symmetry is one of the three known⁷ internal symmetries (i.e., those which do not affect the field u) of the eigenvalue problem (19). In the ψ -coordinate system we write it as $\psi_t = V_0^{(-2)}[\psi] = -\frac{1}{2} \psi$ (the numeric factor is only for simplicity). In the cited reference it is shown that all negative vectors and the internal symmetries span a loop algebra over $SL(2, \mathbb{R})$.

J. Nonlocal Constants of the Motion for KdV

As a final result, we will construct explicitly three new sets of constants of the motion for the KdV equation, starting from the nonlocal objects we have obtained. We denote $\psi = \psi(x, t; \lambda = 0)$ for simplicity from here on, unless explicitly stated. Consider the action principle (25): the term which multiplies the velocity ψ_t is the mapping of the Lagrangian 1-form $P^{(-1)}[u]$ into ψ -coordinates:

$$\mathcal{P}^{(-1)}[\psi](x, t) = \frac{1}{16 \psi^3} (D^{-1} \psi^2).$$

On the other hand, from theorem 7 we have $\mathcal{L}P^{(-1)}_{V_2} = \delta K^{(-1)}$, where $K^{(-1)} = \frac{3}{16} \int_{x_-}^{x_+} dx \frac{\psi_x^2}{\psi^2}$. Consider now the 0-form

$$H^{(-2)}[\psi] \equiv -16 i_{V_0^{(-2)}} P^{(-1)} = - \int_{x_-}^{x_+} dx \frac{1}{\psi^2} (D^{-1} \psi^2)$$

or, more concisely, $H^{(-2)}[\psi] = \int_{x_-}^{x_+} dx \psi \bar{\psi}$.

We use Leibnitz rule to show that this is a constant of motion for the KdV equation, which in ψ -coordinates reads $V_2[\psi] = -\psi_{xxx} + 3\frac{\psi_x\psi_{xx}}{\psi}$: as $\mathcal{L}_{V_2}V_0^{(-2)}[\psi] = 0$, we find: $\mathcal{L}_{V_2}H^{(-2)}[\psi] = \mathcal{L}_{V_0^{(-2)}}K^{(-1)}[\psi] = 0$, where the last equality comes from the fact that $K^{(-1)}[\psi]$ is invariant under scaling of ψ .

Thus $H^{(-2)}[\psi]$ is a nonlocal constant of the motion for the KdV flow. But if we recall that the fields $\psi, \bar{\psi}$ are solutions of the linear problem (19), and that u does not change if these fields are replaced by other arbitrary linear combinations, we get indeed three constants of the motion: $H^{(-1)}[\psi] = \int_{x_-}^{x_+} dx \psi^2$, $H^{(-2)}[\psi] = \int_{x_-}^{x_+} dx \psi \bar{\psi}$, $H^{(-3)}[\psi] = \int_{x_-}^{x_+} dx \bar{\psi}^2$.

There is a reference that supports this construction: in[?], in the context of the eigenvalue “Schrödinger” problem (19), the author assumes that the total probability (here denoted by $H^{(-1)}[\psi]$) is equal to 1. But it is indeed a constant of motion of its own! Moreover, these are indeed special cases ($\lambda = 0$) of more general constants of motion. Along the same lines, we get three families, parametrized by the eigenvalue λ :

$$\begin{aligned} H^{(-1)}(\lambda)[\psi] &= \int_{x_-}^{x_+} dx \psi(\lambda)^2, \\ H^{(-2)}(\lambda)[\psi] &= \int_{x_-}^{x_+} dx \psi(\lambda) \bar{\psi}(\lambda), \\ H^{(-3)}(\lambda)[\psi] &= \int_{x_-}^{x_+} dx \bar{\psi}(\lambda)^2. \end{aligned} \tag{26}$$

These are real new constants (indeed they contain, in their Taylor series around $\lambda = 0$, the constants of the motion from theorem 9). In order to evaluate them explicitly, take, for example, successive derivatives of the first one with respect to λ , evaluate at $\lambda = 0$ and use equation (21). We get:

$$Q^{(-1;n)}[\psi] \equiv \int_{x_-}^{x_+} dx \psi L^{-n} \psi, \quad n = 0, \dots, \infty,$$

and we see they are increasingly nonlocal constants of the motion.

It is worth to mention that these nonlocal constants, when mapped to the coordinate system in which the KdV equation maps into the Harry–Dym equation (see[?]), reproduce the results obtained independently in a recent work[?], and add three more constants to the Harry–Dym equation: the mappings of the nonlocal constants of motion (26) for $\lambda = 0$ into the Harry–Dym equation $\omega_t = (\omega^{-1/2})_{sss}$ for the field $\omega(s, t)$, are $H^{(-1)}[\omega] = \int_{s_-}^{s_+} ds \omega$, $H^{(-2)}[\omega] = \int_{s_-}^{s_+} ds s \omega$, $H^{(-3)}[\omega] = \int_{s_-}^{s_+} ds s^2 \omega$.

V. RESULTS FOR OTHER KDV POSITIVE AND NEGATIVE EQUATIONS

A. Some Positive, Negative and Internal Vectors as known Integrable Equations

We write the internal vectors after transformation to Schwartzian coordinates, defined by $\zeta_x(x, t; \lambda) = \psi(x, t; \lambda)^{-2}$. The internal vectors in ζ -coordinates are $V_0^{(-1)}[\zeta] = 1$, $V_0^{(-2)}[\zeta] = \zeta$, $V_0^{(-3)}[\zeta] = \zeta^2$.

The vectors $V_0^{(-3)}$, $(V_0^{(-3)} - V_1^{(-1)})/2$, and V_2 give the evolution equations $\zeta_t = \zeta^2$, $\zeta_t = \frac{1}{2} \zeta^2 - \frac{1}{8} \frac{\zeta_x \lambda}{\zeta_x}$, and $\zeta_t = 6 \lambda \zeta_x + \frac{3 \zeta_{xx}^2}{2 \zeta_x} - \zeta_{xxx}$, where the fields are evaluated at $\lambda = 0$. The last of these equations is a special case of the Krichever–Novikov equation,[?] and the first and the second equations, via the transformation $z = \ln(2 \zeta_x)$, may be mapped to the Liouville equation $z_{xt} = \exp z$, and the ShG equation $z_{xt} = \sinh z$. For completeness, we just mention that the associated Camassa–Holm equation[?] and the Hunter–Zheng equation[?] are obtainable from the negative vectors $V_1^{(-2)}$ and $V_1^{(-1)}$, respectively, via suitable coordinate transformations, and that the Harry–Dym equation,[?] just like the above case of the Krichever–Novikov equation, is a mapping of the KdV equation.

B. Action Principles for the Sinh–Gordon Equation

As a representative of the extension of the results on action principles for equations in the negative hierarchies, we work out some examples for the ShG equation. The results in this subsection are new up to our knowledge, except when it is explicitly stated. We will work in the z -coordinate system, where the ShG equation is $z_t = V_{\text{ShG}}[z] = D^{-1} \sinh z$.

1. Pure ShG equation: Symplectic matrix $\Sigma^{(0)}$

We look for a standard Lagrangian pair for the ShG equation of the form $(P^{(M)}[z]; K^{(M; \text{ShG})}[z])$, where $\mathcal{P}^{(M)}[z] = -\frac{1}{32} z_x$ is the mapping of $\mathcal{P}^{(M)}[u]$ to z -coordinates. The symplectic 2-form $\Sigma^{(0)}[z] = \frac{1}{16} D$ has only one vector in the kernel, namely $V_0^{(-2)}[z] = 4$. On the other hand, the standard Lagrangian 0-form solves $\frac{\delta}{\delta z} K^{(M; \text{ShG})}[z] = \mathcal{L}_{V_{\text{ShG}}} P^{(M)}[z] = -\frac{1}{32} (\sinh z - z \cosh z)$. We get after integration the usual action principle for the ShG equation:[?]

$$S[z(x, t)] = \frac{1}{32} \int_{t_-}^{t_+} dt dx (-z_x z_t - 2 \cosh z) ,$$

and the Euler–Lagrange equations are simply $z_t = D^{-1} \sinh z + \theta_1$, where θ_1 is arbitrary.

2. ShG equation deformed with first Positive Vector: Symplectic matrix $\Sigma^{(-1)}$

The next negative 1-form, $P^{(-1)}$, reads $\mathcal{P}^{(-1)}[z](x, t) = -\frac{1}{32} e^z (D^{-1} e^{-z})$. The associated symplectic 2-form is $\Sigma^{(-1)}[z] = -\frac{1}{32} (e^z D^{-1} e^{-z} + e^{-z} D^{-1} e^z)$, which inherits the kernel (generated from $V_1^{(1)}[z] = -z_x$) from that in the u -coordinate system only for special boundary conditions: defining the boundary terms $\bar{f} \equiv f_+ + f_-$, and $f_{\pm} = f(x_{\pm})$, the expression

$$\Sigma^{(-1)}[z] \cdot V_1^{(1)}[z] = \frac{1}{64} (e^z \overline{e^{-z}} - e^{-z} \overline{e^z})$$

is zero only for boundary conditions $z_+ = z_- + i\pi(2n + 1)$, $n \in \mathbb{Z}$.

For other boundary conditions, however, this Lagrange bracket has no kernel, which will show up in the variational principle for the ShG vector by the fact that the Euler–Lagrange equations get deformed by a factor of the vector $V_1^{(1)}[z]$, which is not arbitrary: it depends on the boundary conditions used for the z -coordinates.

In the generic case when $\overline{e^z} \neq 0$ (invertible symplectic 2-form $\Sigma^{(-1)}$), the action principle is explicitly

$$S[z(x, t)] = \int_{t_-}^{t_+} dt \left[-\frac{1}{32} \int_{x_-}^{x_+} dx e^z (D^{-1} e^{-z}) (z_t - D^{-1} \sinh z) + K^{(-1; \text{ShG})}[z] \right],$$

and the Euler–Lagrange equations are

$$\frac{1}{32} (e^z D^{-1} e^{-z} + e^{-z} D^{-1} e^z) (z_t - D^{-1} \sinh z + \theta[A_+, A_-] z_x) = 0$$

or, equivalently,

$$z_t - D^{-1} \sinh z + \theta[A_+, A_-] z_x = 0,$$

where $K^{(-1; \text{ShG})}[z] = \int_{x_-}^{x_+} dx e^z (D^{-1} e^{-z})^2 / 128 + F[A_+, A_-]$, $A_{\pm} \equiv \int_{x_-}^{x_+} dx e^{\pm z}$, and θ, F solve the equation:

$$\delta F[A_+, A_-] = -\theta[A_+, A_-] (\overline{e^{-z}} \delta A_+ + \overline{e^z} \delta A_-) + \frac{1}{8} A_- (A_- - 2 A_+) \delta A_+. \quad (27)$$

There are many solutions of the above equation for a given set of boundary conditions on the limiting values of z_{\pm} , so we discuss, as examples, only two representative, nonintersecting cases of boundary conditions, for which the symplectic 2-form $\Sigma^{(-1)}$ is invertible:

(i) $z_+ = -z_- + i\pi(2n)$, $n \in \mathbb{Z}$; $\cosh z_+ \neq 0$.

A solution of equation (27) is $\theta = -A_+ A_- / 4 \overline{e^z}$, $F = A_+ A_-^2 / 512$, which is well defined because of the boundary conditions used.

(ii) $z_+ = -z_- + i\pi(2n+1)$, $n \in \mathbb{Z}$; $\sinh z_+ \neq 0$.

In this case, a solution of equation (27) is

$$\theta = -(A_+^2 - A_-^2) / 8 \overline{e^z}, \quad F = -\left(A_+^2 A_- - \frac{1}{3}(A_+^3 + A_-^3)\right) / 512.$$

The usual constant of the motion for the ShG equation, $H[z] = \int_{x_-}^{x_+} dx \cosh z$, works in this case also: under the boundary conditions used, we get

$$\dot{H}[z] = \int_{x_-}^{x_+} dx \sinh z (D^{-1} \sinh z - \theta z_x) = \theta \overline{\cosh z} = 0.$$

C. Alternative Lax Pairs and Constants of the Motion for Negative Equations

In reference⁷, the authors find alternative Lax pairs for the KdV equation (as well as for every evolution equation in the KdV positive hierarchy) by making no ansatz: they just use the evolution equation and the hereditary operator.

We present a similar construction, this time for the negative vectors. By so doing we are answering an open question in reference⁷.

As it is shown in⁷, given an evolution equation $u_t = V[u]$, and a recursion operator R for V (i.e., $\mathcal{L}_V R = 0$), it follows that

$$\frac{D}{Dt} R = [V', R], \quad (28)$$

where V' denotes the Frechet derivative, and the square bracket is the commutator. The above equation defines the alternative Lax Pair (R, V') .

Now, take R as the hereditary operator $R[u]$, and V as the negative vector $V_1^{(-1)}(\lambda)[u] = (\psi(\lambda)^2)_x$, for arbitrary ν, λ . We need to evaluate the Frechet derivative of this vector with respect to the field u . Using the transformation matrix $\frac{\delta \psi}{\delta u} = -2\psi D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2$, where ψ stands for $\psi(\lambda)$ from here on, we get $V_1^{(-1)}(\lambda)'[u] = -4D\psi^2 D^{-1} \frac{1}{\psi^2} D^{-1} \psi^2$. On the other hand, the hereditary operator is

$$R[u] = R(\lambda)[u] - 4\lambda \mathbb{I} = \frac{1}{\psi^2} D\psi^2 D\psi^2 D\frac{1}{\psi^2} D^{-1} - 4\lambda \mathbb{I},$$

and it is also written as $R[u] = D^2 + 8u + 4u_x D^{-1}$. Now we apply the Lax pair equation (28), getting after some rearrangements the operator equation $8u_t + 4u_{xt} D^{-1} = 8(\psi^2)_x + 4(\psi^2)_{xx} D^{-1}$, which implies $u_t = (\psi(\lambda)^2)_x$. Recall this equation contains all the negative vectors in the corresponding negative hierarchy, so that the Lax pair we have presented indeed works for all vectors in that hierarchy. Similarly, for the other two negative hierarchies we get the Lax pairs (R, B_2) and (R, B_3) , with

$$\begin{aligned} B_2 &= V_1^{(-2)}(\lambda)'[u] = -4D\psi^2 D^{-1} \bar{\psi} \psi^{-1} D \bar{\psi} \psi^{-1} D^{-1} (\bar{\psi})^{-2} D^{-1} \psi \bar{\psi}, \\ B_3 &= V_1^{(-3)}(\lambda)'[u] = -4D\bar{\psi}^2 D^{-1} (\bar{\psi})^{-2} D^{-1} \bar{\psi}^2. \end{aligned}$$

In this way, we may construct an infinite number of constants of the motion for the negative vectors, from Adler traces of positive, semi-integer powers of the Nijenhuis operator R : these are just the usual (local) constants of the motion for the KdV equation.⁷ A natural conjecture is that Adler traces of positive, semi-integer powers of R^{-1} will give our nonlocal constants for KdV defined in equation (26). If that is true, we could infer that the nonlocal constants of the motion for KdV should also work for the negative vectors, which can be explicitly checked. We present the results only for the hierarchy $V_1^{(-1)}(\lambda)$, because Lie derivatives of the results along the internal vector $V_0^{(-3)}$ map the objects into similar ones for the other two negative hierarchies:

- (i) Conserved currents: defining the boundary term $\widehat{f} \equiv f(x_+) - f(x_-)$, the integral $H^{(-1)}(\nu)[\psi] = \int_{x_-}^{x_+} dx \psi(\nu)^2$ solves

$$\mathcal{L}_{V_1^{(-1)}(\lambda)} H^{(-1)}(\nu)[\psi] = -\overbrace{E(\nu, \lambda)[\psi]},$$

$$\text{where } E(\nu, \lambda)[\psi] \equiv \frac{\psi(\lambda) \psi_x(\nu) - \psi(\nu) \psi_x(\lambda)}{\lambda - \nu}.$$

- (ii) Constants of the motion: the expression

$$G^{(-1)}(\nu, \lambda)[\psi, t] = \int_{x_-}^{x_+} dx \psi(\nu)^2 + t \overbrace{\frac{E(\nu, \lambda)[\psi]^2}{1 - t E(\lambda, \lambda)[\psi]^2}}$$

is a constant of the motion for the flow $V_1^{(-1)}(\lambda)$.

These constants and currents are infinite in number and work for every vector in the respective negative hierarchy, because λ and ν are arbitrary. By considering their Taylor expansion around $\lambda = \nu = 0$, the explicit expression for the constants and conserved currents for each negative vector is easily worked out using equations (21) and (18).

VI. CONCLUSION

The Lagrangian point of view determines a unifying scheme for the study of integrable equations belonging to hierarchies related to hereditary operators. For all evolution vectors in these hierarchies, nonlocal symmetries, Lax pairs, constants of the motion, conserved currents and an infinite ladder of action principles all come out in a constructive, explicit way from the same structure. Moreover, new equations, which are mixed or deformed versions of known integrable equations, arise as the Euler–Lagrange equations of the action principles obtained. As an example, we apply this scheme to the KdV equation, and the results are directly mappable to other related equations in the positive KdV hierarchies (e.g., Harry–Dym and a special case of Krichever–Novikov equations) as well as in the negative KdV hierarchies (e.g., Sinh–Gordon, Liouville, Camassa–Holm and Hunter–Zheng equations): in particular, we obtain a new nonlocal action principle for the Sinh–Gordon equation which leads to a deformed version of this equation, and an infinite number of nonequivalent, nonlocal action principles for KdV, possessing time–reparametrization invariance, are explicitly found. The construction of alternative Lax pairs for negative equations arises naturally, without any ansatz, from this scheme, and it is shown that negative equations essentially share the constants of the motion (local as well as nonlocal) for the KdV equation.

VII. ACKNOWLEDGEMENTS

M.B. wishes to mention that this work was done in part during a research stay at the Relativity Center, The University of Austin, Texas, U.S.A., where M.B. was invited by Prof. L. C. Shepley; M.B. is deeply grateful to him, the professors and students in the Center for their support and the following people whose comments and suggestions were fundamental for his understanding of the topics involved in the present work: P. J. Morrison, C. deWitt–Morette, C. Valls, J. Zanelli, C. Teitelboim, R. Troncoso.

Finally, M.B. acknowledges the financial support from a Fundación Andes Grant for Doctoral Studies, and a Conicyt Grant for Thesis Completion.